Flexible global generalized Hessenberg methods for linear systems with multiple right-hand sides

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Outline

1 Problem, applications and methods

2 Global generalized Hessenberg method (GI-GH)

3 Flexible preconditioning

4 Flexible global generalized Hessenberg method (FGI-GH)

5 Examples & Remarks
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3. Flexible preconditioning
4. Flexible global generalized Hessenberg method (FGI-GH)
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1 Problem, applications and methods

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5 Examples & Remarks
We consider the solution of large and sparse linear system with multiple right-hand sides of the form

\[ AX = B, \quad (1.1) \]

where \( A \) is an \( n \times n \) nonsingular matrix, and \( X, B \in \mathbb{R}^{n \times s} \) with usually \( s \ll n \).
Applications

- Chemistry
- Electromagnetic wave scattering
- Structures and Control
- Radar cross section analysis
Existing methods for solving (1.1)

**Block methods**
- Block CG and BiCG [O’Leary/LAA1980]
- Block QMR [Freund etc./LAA1997]
- Block BiCGSTAB [Guennouni etc./ETNA2003]
- Block IDR(s) [Du etc./JCAM2011]
  
  *More effective for dense linear system.*

**Seed methods**
- [Smith/IEEE1989], [Chan etc./SIAM1999], [Abdel-Rehim/NLAA2013]

**Global methods**
- Global iteration [Saad/1996]
- Global FOM and GMRES [Jbilou etc./ANM1999]
- Global Hessenberg and CMRH [Heyouni/NA2001]
- Global BiCG and BiCGSTAB [Jbilou etc./ETNA2005]
- Global SCD [Gu etc./AMC2007]
  
  *More effective for sparse linear system.*
The **matrix Krylov subspace** $\mathcal{K}_m(A, V)$ is spanned by $V$, $AV$, $\cdots$, $A^{m-1}V$, or equivalently, for any $W \in \mathcal{K}_m(A, V)$, we have

$$W = \sum_{i=1}^{m} \alpha_i A^{i-1}V, \quad (2.1)$$

where $V \in \mathbb{R}^{n \times s}$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \cdots, m$. Associated with the matrix Krylov subspace is the product $\ast$ defined by

$$\mathcal{V}_m \ast x = \sum_{i=1}^{m} x_i V_i, \quad (2.2)$$

where $\mathcal{V}_m = [V_1, \cdots, V_m] \in \mathbb{R}^{n \times ms}$ and $x = [x_1, \cdots, x_m]^T \in \mathbb{R}^m$. 

In addition, if $H \in \mathbb{R}^{m \times m}$ and $y \in \mathbb{R}^m$, then we have

$$
\mathcal{V}_m \ast H = [\mathcal{V}_m \ast H_{.,1}, \cdots, \mathcal{V}_m \ast H_{.,m}], \quad (2.3)
$$

$$
\mathcal{V}_m \ast (x + y) = (\mathcal{V}_m \ast x) + (\mathcal{V}_m \ast y), \quad (2.4)
$$

$$
(\mathcal{V}_m \ast H) \ast x = \mathcal{V}_m \ast (Hx). \quad (2.5)
$$
The **Gl-GH process** [Heyouni etc.//NA2005] generates a matrix basis 
\[ \text{span}\{V_1, \cdots, V_m\} \] of the matrix Krylov subspace \( \mathcal{K}_m(A, R_0) \) through the relations

\[
V_1 = \frac{R_0}{\beta} \quad \text{and} \quad (\bar{H}_m)_{i+1,i} V_{i+1} = AV_i - \sum_{k=1}^{i} (\bar{H}_m)_{k,i} V_k, \quad (2.6)
\]

where the initial residual \( R_0 = B - AX_0 \), \( \beta \) and \( (\bar{H}_m)_{i+1,i} \) are nonzero scaling factors for \( i = 1, \cdots, m \).
Let $Y_1, \cdots, Y_m$ be linearly independent matrices, where $Y_i \in \mathbb{R}^{n \times s}$ for $i = 1, \cdots, m$. The scalars $(\bar{H}_m)_{j,i}$ in (2.6) are obtained by imposing the orthogonality condition

$$V_{i+1} \perp_F Y_1, \cdots, Y_i, \quad i = 1, \cdots, m.$$  \hspace{1cm} (2.7)

Using (2.6) and (2.7), we have

$$(\bar{H}_m)_{j,i} = \frac{(Y_j, U)_F}{(Y_j, V_j)_F} = \frac{\text{tr}(Y_j^T U)}{\text{tr}(Y_j^T V_j)},$$

where $U = AV_i - \sum_{k=1}^{j-1} (\bar{H}_m)_{k,i} V_k$ for $j = 1, \ldots, i$. 
Alg. 1: GI-GH Process (with fixed preconditioning).

1: $\beta = \|V\|$, $V_1 = V / \beta$;
2: for $i = 1, \cdots, m$ do
3: $Z_i = M^{-1}V_i$; \% inner process with a fixed preconditioner $M$
4: $U = AZ_i$;
5: for $j = 1, \cdots, i$ do
6: $(\bar{H}_m)_{j,i} = \text{tr}(Y_j^T U) / \text{tr}(Y_j^T V_j)$; $U = U - (\bar{H}_m)_{j,i} V_j$;
7: end for
8: $(\bar{H}_m)_{i+1,i} = \|U\|$; $V_{i+1} = U / (\bar{H}_m)_{i+1,i}$;
9: end for
Alg. 2: Gl-GH method

Gl-GH: the global generalized Hessenberg method with fixed preconditioning.

1: Choose $X_0$ and compute $R_0 = B - AX_0$. Set $\beta = \|R_0\|, V_1 = R_0/\beta$;
2: Generate the block matrix $\mathcal{V}_m = [V_1, \cdots, V_m]$ from Algorithm 10. Update $X_m = X_0 + M^{-1}\mathcal{V}_m \ast y_m$, where $y_m = \arg \min_{y \in \mathbb{R}^m} \|\beta e_1 - \bar{H}_m y\|_2$.
3: If converged then stop; otherwise set $X_0 = X_m$ and goto line 1.
Flexible right preconditioning

We now consider a right preconditioning for the original linear system (1.1), namely,

\[ AM^{-1}(MX) = B, \quad (3.1) \]

where \( M \) is an appropriate preconditioner.

Idea([Saad/SIAM1993]): Use flexible preconditioners \( M_i \) (instead of fixed \( M \)) here!
Alg. 3: FGI-GH: the flexible global generalized Hessenberg method [Zhang&Gu/JCAM2014]

1: Choose $X_0$ and $m$.
2: Compute $R_0 = B - AX_0$. Set $\beta = \|R_0\|$, $V_1 = R_0/\beta$;
3: \textbf{for} $i = 1, \cdots, m$ \textbf{do}
4: \hspace{1em} $Z_i = M_i^{-1}V_i$; \hspace{1em} \% inner process with a flexible preconditioner $M_i$
5: \hspace{1em} $U = AZ_i$;
6: \hspace{1em} \textbf{for} $j = 1, \cdots, i$ \textbf{do}
7: \hspace{2em} $(\bar{H}_m)_{j,i} = \text{tr}(Y_j^T U) / \text{tr}(Y_j^T V_j)$; $U = U - (\bar{H}_m)_{j,i} V_j$;
8: \hspace{1em} \textbf{end for}
9: \hspace{1em} $(\bar{H}_m)_{i+1,i} = \|U\|$; $V_{k+1} = U / (\bar{H}_m)_{i+1,i}$;
10: \textbf{end for}
11: Form $\mathcal{Z}_m = [Z_1, \cdots, Z_m]$ by solving an inner system at line 4.
    Update $X_m = X_0 + \mathcal{Z}_m \ast y_m$, where
    $y_m = \arg \min_{y \in \mathbb{R}^m} \| \beta e_1 - \bar{H}_m y \|_2$.
12: If converged then stop; otherwise set $X_0 = X_m$ and goto line 2.
Relations from Alg. 3

It follows from Alg. 3 that

\[ A\mathcal{X}_m = \mathcal{V}_{m+1} \ast \bar{H}_m, \quad (4.1) \]

where \( \mathcal{V}_{m+1} = [V_1, \ldots, V_{m+1}] \) and \( \mathcal{Z}_m = [Z_1, \ldots, Z_m] \). Suppose that a second iterative solver is applied to solve the inner system \( M_i Z_i = V_i \) in Alg. 3. Then we consider the following relation

\[ Z_i = M_i^{-1} V_i = M^{-1} V_i + G_i, \quad (4.2) \]

where \( G_i \) is the error matrix related to the inner solver at step \( i \). The \( m \)-th residual of FGI-GH can be expressed as

\[ R_m = R_0 - A\mathcal{X}_m \ast y_m \quad (4.3) \]

\[ = R_0 - (\mathcal{V}_{m+1} \ast \bar{H}_m) \ast y_m \quad (4.4) \]

\[ = \mathcal{V}_{m+1} \ast (\beta e_1 - \bar{H}_m y_m). \quad (4.5) \]
Alg. 4: FGI-GMRES

1: Choose $X_0$ and $m$.
2: Compute $R_0 = B - AX_0$. Set $\beta = \|R_0\|_F$ and $V_1 = R_0/\beta$;
3: for $i = 1, \cdots, m$ do
4: $Z_i = M_i^{-1}V_i$; % inner process with a flexible preconditioner $M_i$
5: $U = AZ_i$;
6: for $j = 1, \cdots, i$ do
7: $(\bar{H}_m)_{j,i} = \text{tr}(V_j^T U)$; $U = U - (\bar{H}_m)_{j,i}V_j$;
8: end for
9: $(\bar{H}_m)_{i+1,i} = ||U||_F$; $V_{k+1} = U/(\bar{H}_m)_{i+1,i}$;
10: end for
11: Form $Z_m = [Z_1, \cdots, Z_m]$ by solving an inner system at line 4. Update $X_m = X_0 + Z_m * y_m$, where $y_m = \text{arg min}_{y \in \mathbb{R}^m} \|\beta e_1 - \bar{H}_m y\|_2$.
12: If converged then stop; otherwise set $X_0 = X_m$, $R_0 = R_m$ and goto line 2.
Alg. 5: FGI-CMRH: the flexible global CMRH method.

1: Choose $X_0$ and $m$.
2: Compute $R_0 = B - AX_0$. Determine $(R_0)_{i_0,j_0} = \max_{1 \leq i \leq n, 1 \leq j \leq s} \{|(R_0)_{i,j}|\}$. Set $\beta = (R_0)_{i_0,j_0}$, $V_1 = R_0/\beta$, $p_{1,1} = i_0$ and $p_{1,2} = j_0$;
3: for $i = 1, \ldots, m$ do
4: $Z_i = M_i^{-1}V_i$; % inner process with a flexible preconditioner $M_i$
5: $U = AZ_i$;
6: for $j = 1, \ldots, i$ do
7: $(\overline{H}_m)_{j,i} = U_{p_j,1}p_j,2; U = U - (\overline{H}_m)_{j,i}V_j$;
8: end for
9: Determine $i_0, j_0$ such that $U_{i_0,j_0} = \max_{1 \leq i \leq n, 1 \leq j \leq s} \{|U_{i,j}|\}$.
10: $(\overline{H}_m)_{i+1,i} = U_{i_0,j_0}; V_{i+1} = U/(\overline{H}_m)_{i+1,i}; p_{i+1,1} = i_0$ and $p_{i+1,2} = j_0$.
11: end for
12: Form $Z_m = [Z_1, \ldots, Z_m]$ solving an inner system at line 4. Update $X_m = X_0 + Z_m * y_m$, where $y_m = \arg \min_{y \in \mathbb{R}^m} \|\beta e_1 - \overline{H}_m y\|_2$.
13: If converged then stop; otherwise set $X_0 = X_m$ and goto line 2.
## Computational cost

**Table 1:** Total cost per restart with GI-BiCGSTAB as the inner preconditioner for flexible variants.

<table>
<thead>
<tr>
<th>Global process (global method)</th>
<th>Number of operations</th>
</tr>
</thead>
<tbody>
<tr>
<td>global Arnoldi (GI-GMRES)</td>
<td>$2sm(m + 4)n + 2msN$</td>
</tr>
<tr>
<td>global Hessenberg (GI-CMRH)</td>
<td>$sm(m + 3)n + 2msN$</td>
</tr>
<tr>
<td>flexible global Arnoldi (FGI-GMRES)</td>
<td>$2sm(m + 4)n + 2msN + \sum_{i=1}^{m} (2k^{(i)}(10s + 1)n + 4k^{(i)}sN)$</td>
</tr>
<tr>
<td>flexible global Hessenberg (FGI-CMRH)</td>
<td>$sm(m + 3)n + 2msN + \sum_{i=1}^{m} (2k^{(i)}(10s + 1)n + 4k^{(i)}sN)$</td>
</tr>
</tbody>
</table>
Lemma 1

Let \( Z^g_{ih} \) and \( Z^f_{ih} \) be defined in Gl-GH and FGl-GH, respectively. If \( G^f_{ih} \) is the \( i \)-th error matrix defined in (4.2), then

\[
Z^g_{ih} \in S_m, \quad i = 1, \ldots, m, \tag{4.6}
\]

where the subspace \( S_m = \text{span}\{Z^f_{ih}, (M^{-1}A)^j G^f_{ih}\}_{j=0,\ldots,m-1} \).
Theorem 2

Let $R_{fgh}^m$ and $R_{gh}^m$ be the $m$-th residuals derived from FGl-GH and Gl-GH respectively. Define the block error matrix $G_{fgh}^m = [G_{1}^{fgh}, \cdots, G_{m}^{fgh}]$, where $G_{i}^{fgh}$ are defined in (4.2) for $i = 1, \cdots, m$. Then there exist $d_{j}^{fgh} \in \mathbb{R}^m$ s.t.

$$
\| R_{fgh}^m \|_F \leq \kappa_F(\gamma_{m+1}^{fgh})(\| R_{gh}^m \|_F + \sum_{j=0}^{m-1} \| A(M^{-1}A)^j g_{m}^{fgh} \ast d_{j}^{fgh} \|_F),
$$

(4.7)

where the product $\ast$ is defined in (2.2), $\kappa_F(\gamma_{m+1}^{fgh}) = \| (\gamma_{m+1}^{fgh})^\dagger \|_F \| \gamma_{m+1}^{fgh} \|_F$, $\gamma_{m+1}^{fgh}$ is defined in (4.1), and $(\gamma_{m+1}^{fgh})^\dagger$ is the left inverse of $\gamma_{m+1}^{fgh}$. 
Corollary 3

Let $R_{m}^{fgc}$ and $R_{m}^{gc}$ be the $m$-th residuals derived from FGI-CMRH and GI-CMRH respectively. Define $G_{m}^{fgc} = [G_{1}^{fgc}, \cdots, G_{m}^{fgc}]$, where $G_{i}^{fgc}$ are defined in (4.2) for $i = 1, \cdots, m$. Then there exist $d_{j}^{fgc} \in \mathbb{R}^{m}$ such that

$$
\| R_{m}^{fgc} \|_{F} \leq \kappa_{F}(\gamma_{m+1}^{fgc}) (\| R_{m}^{gc} \|_{F} + \sum_{j=0}^{m-1} \| A(M^{-1}A)j G_{m}^{fgc} \ast d_{j}^{fgc} \|_{F}).
$$
Corollary 4

Let $R_{m}^{fgg}$ and $R_{m}^{gg}$ be the $m$-th residuals derived from FGl-GMRES and Gl-GMRES respectively. Define $G_{m}^{fgg} = [G_{1}^{fgg}, \cdots, G_{m}^{fgg}]$, where $G_{i}^{fgg}$ are defined in (4.2) for $i = 1, \cdots, m$. Then there exist $d_{j}^{fgg} \in \mathbb{R}^{m}$ such that

$$\| R_{m}^{fgg} \|_{F} \leq \kappa_{F}(\gamma_{m+1}^{fgg})(\| R_{m}^{gg} \|_{F} + \sum_{j=0}^{m-1} \| A(M^{-1}A)^{j} G_{m}^{fgg} \ast d_{j}^{fgg} \|_{F}).$$
Lemma 5 (Heyouni/NA2001)

If $R_{gm}^{gc}$ and $R_{gm}^{gg}$ denote respectively the $m$-th residuals of Gl-CMRH and Gl-GMRES, then

$$\| R_{gm}^{gc} \|_F \leq \kappa_F (\gamma_{m+1}^{gc}) \| R_{gm}^{gg} \|_F.$$
Lemma 6

Suppose that $\mathcal{L}_m^{fgc}$ and $\mathcal{L}_m^{fgg}$ are given by (4.1), and that $G_i^{fgc}$ and $G_i^{fgg}$ are defined in (4.2). Then

$$Z_i^{fgg} \in \mathcal{P}_m, \ i = 1, \cdots, m,$$

(4.8)

where the subspace

$$\mathcal{P}_m = \text{span}\{Z_i^{fgc}, (M^{-1}A)^j G_i^{fgc}, (M^{-1}A)^j G_i^{fgg}\}_{j=0, \cdots, m-1}.$$

Theorem 7

Let $G_m^{fgc} = [G_{1}^{fgc}, \ldots, G_{m}^{fgc}]$ and $G_m^{fgg} = [G_{1}^{fgg}, \ldots, G_{m}^{fgg}]$, where $G_i^{fgc}$ and $G_i^{fgg}$ are defined in (4.2). Then there exist $\hat{d}_j^{fgc}, \hat{d}_j^{fgg} \in \mathbb{R}$ such that

$$\| R_m^{fgc} \|_F \leq \kappa_F (\gamma_{m+1}) \| R_m^{fgg} \|_F + \sum_{j=0}^{m-1} \| A(M^{-1}A)^j G_m^{fgc} \ast \hat{d}_j^{fgc} \|_F$$

$$+ \sum_{j=0}^{m-1} \| A(M^{-1}A)^j G_m^{fgg} \ast \hat{d}_j^{fgg} \|_F.$$
Corollary 8

If $g_{m}^{fgc} = 0$, then the relation between GI-CMRH and FGI-GMRES is given by

$$\| R_{m}^{gc} \|_F \leq \kappa_F(\gamma_{m+1}^{gc})(\| R_{m}^{fgg} \|_F + \sum_{j=0}^{m-1} \| A(M^{-1}A)^j g_{m}^{fgg} \diamond d_{j}^{fgg} \|_F).$$

Corollary 9

If $g_{m}^{fgg} = 0$, then the relation between FGI-CMRH and GI-GMRES is given by

$$\| R_{m}^{fgc} \|_F \leq \kappa_F(\gamma_{m+1}^{fgc})(\| R_{m}^{gg} \|_F + \sum_{j=0}^{m-1} \| A(M^{-1}A)^j g_{m}^{fgc} \diamond d_{j}^{fgc} \|_F).$$
### Example 1

**Table 2:** Numerical results for Example 1.

<table>
<thead>
<tr>
<th>Method</th>
<th>Matrix $A$</th>
<th>Restarts</th>
<th>CPU time (Sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GI-Hess</td>
<td>$\text{sherman1}$</td>
<td>144</td>
<td>19.83</td>
</tr>
<tr>
<td>GI-GMRES</td>
<td>$n = 1000$</td>
<td>515</td>
<td>194.9</td>
</tr>
<tr>
<td>WGI-GMRES</td>
<td>$s = 30, m = 20$</td>
<td>147</td>
<td>76.9</td>
</tr>
<tr>
<td>FGI-GMRES</td>
<td>$max_{out} = 1500$</td>
<td>2</td>
<td>7.9</td>
</tr>
<tr>
<td>GI-CMRH</td>
<td>$max_{in} = 20$</td>
<td>395</td>
<td>56.6</td>
</tr>
<tr>
<td>WGI-CMRH</td>
<td>$tol_{out} = 1e−13$</td>
<td>307</td>
<td>51.1</td>
</tr>
<tr>
<td>FGI-CMRH</td>
<td>$tol_{in} = 1e−2$</td>
<td>3</td>
<td>10.5</td>
</tr>
<tr>
<td>GI-Hess</td>
<td>$\text{orsirr1}$</td>
<td>274</td>
<td>11.3</td>
</tr>
<tr>
<td>GI-GMRES</td>
<td>$n = 1030$</td>
<td>867</td>
<td>80.0</td>
</tr>
<tr>
<td>WGI-GMRES</td>
<td>$s = 10, m = 20$</td>
<td>259</td>
<td>27.7</td>
</tr>
<tr>
<td>FGI-GMRES</td>
<td>$max_{out} = 1500$</td>
<td>5</td>
<td>4.2</td>
</tr>
<tr>
<td>GI-CMRH</td>
<td>$max_{in} = 20$</td>
<td>1000</td>
<td>42.4</td>
</tr>
<tr>
<td>WGI-CMRH</td>
<td>$tol_{out} = 1e−12$</td>
<td>947</td>
<td>45.3</td>
</tr>
<tr>
<td>FGI-CMRH</td>
<td>$tol_{in} = 1e−1$</td>
<td>7</td>
<td>7.56</td>
</tr>
</tbody>
</table>
Figure 1: Convergence history for sherman1 in Example 1.
### Example 2

**Table 3: Results for Example 2.** “–” stands for no convergence.

<table>
<thead>
<tr>
<th>Method</th>
<th>Matrix A</th>
<th>Restarts</th>
<th>CPU time (Sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gl-Hess</td>
<td><strong>rdb2048l</strong></td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Gl-GMRES</td>
<td>( n = 2048 )</td>
<td>213</td>
<td>20.3</td>
</tr>
<tr>
<td>WGI-GMRES</td>
<td>( s = 20, m = 8 )</td>
<td>230</td>
<td>29.2</td>
</tr>
<tr>
<td>FGI-GMRES</td>
<td>( max_{out} = 1500 )</td>
<td>5</td>
<td>10.6</td>
</tr>
<tr>
<td>Gl-CMRH</td>
<td>( max_{in} = 20 )</td>
<td>450</td>
<td>26.0</td>
</tr>
<tr>
<td>WGI-CMRH</td>
<td>( tol_{out} = 1e - 14 )</td>
<td>976</td>
<td>69.9</td>
</tr>
<tr>
<td>FGI-CMRH</td>
<td>( tol_{in} = 1e - 1 )</td>
<td>7</td>
<td>16.7</td>
</tr>
<tr>
<td>Gl-Hess</td>
<td><strong>rdb3200l</strong></td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Gl-GMRES</td>
<td>( n = 3200 )</td>
<td>202</td>
<td>42.4</td>
</tr>
<tr>
<td>WGI-GMRES</td>
<td>( s = 20, m = 10 )</td>
<td>160</td>
<td>39.6</td>
</tr>
<tr>
<td>FGI-GMRES</td>
<td>( max_{out} = 1500 )</td>
<td>5</td>
<td>18.4</td>
</tr>
<tr>
<td>Gl-CMRH</td>
<td>( max_{in} = 20 )</td>
<td>334</td>
<td>40.2</td>
</tr>
<tr>
<td>WGI-CMRH</td>
<td>( tol_{out} = 1e - 13 )</td>
<td>289</td>
<td>38.6</td>
</tr>
<tr>
<td>FGI-CMRH</td>
<td>( tol_{in} = 1e - 2 )</td>
<td>6</td>
<td>24.5</td>
</tr>
</tbody>
</table>
### Example 3

**Table 4: Results for Example 3.**

<table>
<thead>
<tr>
<th>Method</th>
<th>Matrix</th>
<th>Restarts</th>
<th>CPU time (Sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GI-Hess</td>
<td><strong>pde2961</strong></td>
<td>200</td>
<td>44.0</td>
</tr>
<tr>
<td>GI-GMRES</td>
<td>$n = 2961$</td>
<td>88</td>
<td>38.2</td>
</tr>
<tr>
<td>WGI-GMRES</td>
<td>$s = 30, m = 10$</td>
<td>52</td>
<td>30.6</td>
</tr>
<tr>
<td>FGI-GMRES</td>
<td>$max_{out} = 200$</td>
<td>2</td>
<td>19.7</td>
</tr>
<tr>
<td>GI-CMRH</td>
<td>$max_{in} = 100$</td>
<td>142</td>
<td>31.2</td>
</tr>
<tr>
<td>WGI-CMRH</td>
<td>$tol_{out} = 1e-14$</td>
<td>111</td>
<td>29.3</td>
</tr>
<tr>
<td>FGI-CMRH</td>
<td>$tol_{in} = 1e-1$</td>
<td>2</td>
<td>26.8</td>
</tr>
<tr>
<td>GI-Hess</td>
<td><strong>memplus</strong></td>
<td>184</td>
<td>35.3</td>
</tr>
<tr>
<td>GI-GMRES</td>
<td>$n = 17758$</td>
<td>758</td>
<td>269.6</td>
</tr>
<tr>
<td>WGI-GMRES</td>
<td>$s = 2, m = 20$</td>
<td>157</td>
<td>79.2</td>
</tr>
<tr>
<td>FGI-GMRES</td>
<td>$max_{out} = 1500$</td>
<td>4</td>
<td>17.9</td>
</tr>
<tr>
<td>GI-CMRH</td>
<td>$max_{in} = 20$</td>
<td>695</td>
<td>144.2</td>
</tr>
<tr>
<td>WGI-CMRH</td>
<td>$tol_{out} = 1e-10$</td>
<td>540</td>
<td>122.8</td>
</tr>
<tr>
<td>FGI-CMRH</td>
<td>$tol_{in} = 1e-1$</td>
<td>5</td>
<td>23.6</td>
</tr>
</tbody>
</table>
Figure 2: Convergence history for **memplus** in Example 3.
Remarks

1. The performance of FGl-GH is in general much better than GI-GH; cf., e.g., FGl-GMRES/GI-GMRES or FGl-CMRH/GI-CMRH.

2. FGl-GMRES seems to the best while FGl-CMRH appears to be the second among the listed algorithms.

3. Due to the existence vectors $d_j$ in Thm 2 etc., it is not clear how sharp the corresponding upper bounds are and how can they be applied in designing new algs. in the future.
References


Thank you!